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Abstract
Decompositions of the unitary group $\mathbf{U}(n)$ are useful tools in quantum information theory as they allow one to decompose unitary evolutions into local evolutions and evolutions causing entanglement. Several recursive decompositions have been proposed in the literature to express unitary operators as products of simple operators with properties relevant in entanglement dynamics. In this paper, using the concept of grading of a Lie algebra, we cast these decompositions in a unifying scheme and show how new recursive decompositions can be obtained. In particular, we propose a new recursive decomposition of the unitary operator on N qubits, and give a numerical example.

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1. Introduction

Decompositions of the unitary Lie group $\mathbf{U}(n)$ serve to factorize any element $X_f \in \mathbf{U}(n)$ as a product $X_f = X_1 X_2 \ldots X_m$, where $X_1, X_2, \ldots, X_m$ are (elementary) factors in $\mathbf{U}(n)$. There are several reasons to study such decompositions for unitary evolutions in quantum mechanics. They allow one to analyze the dynamics of a quantum system in terms of simpler, possibly meaningful, factors. In particular, for multipartite systems they allow the identification of the local and entangling parts of a given evolution. In this context one can study entanglement dynamics [1, 2, 11]. From a more practical point of view, they allow one to decompose the task of designing a given evolution, such as a quantum gate, into simpler, readily available dynamics (cf, e.g., [8]). In particular, in multipartite systems few entangling evolutions are typically available. Lie group decompositions are also useful in control problems [3], in the solution of some algebraic problems of interest in quantum information [9] and in the design of the quantum circuits [12, 13]. For these reasons several decompositions have been introduced in recent years [1, 2, 4, 5, 7]. In [1, 2], a decomposition called the *concurrence*

canonical decomposition (CCD) was studied in the context of entanglement theory. The CCD is a way to decompose every unitary evolution on N qubits into a part that does not modify the concurrence on the N qubits, and a part that does. It is a Cartan decomposition in that it corresponds to a symmetric space of $\mathbf{SU}(2^N)$ [6]. In [4], the CCD was further studied and generalized to multipartite systems of arbitrary dimensions. The resulting decomposition was called an *odd–even decomposition* (OED). The OED is a decomposition of unitary evolutions on multipartite systems constructed in terms of decompositions on the single subsystems. *Recursive* decompositions such as the ones in [5] and [7] recursively apply the Cartan decomposition theorem in order to decompose the factors into simpler ones.

The present paper is devoted to recursive decompositions. Using the relation between Cartan decompositions of Lie algebras and Lie algebra gradings, we show that the recursive decompositions of [5] and [7] are a special case of a general scheme from which several other recursive decompositions can be obtained.

The paper is organized as follows. Most of the content of section 2 is background material concerning the basic concepts of Cartan decompositions of Lie groups and algebras, with particular emphasis on decompositions of $\mathbf{U}(n)$. We also describe the main ingredients of the CCD decomposition of [1] and [2]; the OED decomposition of [4]; and the recursive decompositions of Khaneja and Glaser [7], and D'Alessandro and Romano [5]. One extension of the procedure used for the OED decomposition is presented in theorem 2.1. In section 3, we describe gradings of Lie algebras and establish a link between gradings and recursive decompositions. This gives a general method to develop recursive decompositions of $\mathbf{U}(n)$. We show in section 4 how the recursive decompositions of [5] and [7] are special cases of this general procedure and how new recursive decompositions can be obtained. In section 5, we give a numerical example illustrating the calculation of the recursive decompositions described in section 4. Some concluding remarks are presented in section 6.

2. Cartan decompositions of the unitary group

2.1. Cartan decompositions of a Lie algebra

A Cartan decomposition of a semisimple Lie algebra $\mathcal{L}$ is a vector space decomposition

$$\mathcal{L} = \mathcal{K} \oplus \mathcal{P}, \tag{1}$$

where the subspaces $\mathcal{K}$ and $\mathcal{P}$ satisfy the commutation relations

$$[\mathcal{K}, \mathcal{K}] \subseteq \mathcal{K}, \qquad [\mathcal{K}, \mathcal{P}] \subseteq \mathcal{P}, \qquad [\mathcal{P}, \mathcal{P}] \subseteq \mathcal{K}.$$

The pair $(\mathcal{K}, \mathcal{P})$ is called a *Cartan pair* of $\mathcal{L}$. In particular, $\mathcal{K}$ is closed under the Lie bracket and is therefore a Lie subalgebra of $\mathcal{L}$. A Cartan decomposition of a Lie algebra $\mathcal{L}$ induces a decomposition of the connected Lie group associated with $\mathcal{L}$, which we denote by $e^{\mathcal{L}}$. In particular, every element X of $e^{\mathcal{L}}$ can be written as

$$X = KP, \tag{2}$$

where $K \in e^{\mathcal{K}}$ and P is the exponential of an element in $\mathcal{P}$. Since $[\mathcal{P}, \mathcal{P}] \subseteq \mathcal{K}$, any Lie subalgebra contained in $\mathcal{P}$ is necessarily Abelian. A maximal Abelian subalgebra $\mathcal{H}$ contained in $\mathcal{P}$ is called a *Cartan subalgebra*, and the common dimension of all the maximal Abelian subalgebras $\mathcal{H}$ is called the *rank* of the decomposition. Indeed, although the Cartan subalgebra is not unique, it may be shown that two Cartan subalgebras $\mathcal{H}$ and $\mathcal{H}_1$ are conjugate via an element of $e^{\mathcal{K}}$. This means that there exists $S \in e^{\mathcal{K}}$ such that $\mathcal{H} = Ad_S(\mathcal{H}_1)$. Here Ad_S denotes the adjoint map defined as $Ad_S(H) := SHS^{\dagger}$ for $H \in \mathcal{L}$.

Let $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{L}$. One can prove that

$$\mathcal{P} = \bigcup_{S \in \mathrm{e}^{\mathcal{K}}} Ad_S(\mathcal{H})$$

and therefore

$$\exp(\mathcal{P}) = \bigcup_{S \in \mathrm{e}^{\mathcal{K}}} Ad_S(\mathrm{e}^{\mathcal{H}}).$$

It follows that P in (2) has the form $P = SAS^{\dagger}$, with $S \in \mathrm{e}^{\mathcal{K}}$ and $A \in \mathrm{e}^{\mathcal{H}}$. Hence, from (2), each $X \in \mathrm{e}^{\mathcal{L}}$ can be written as

$$X = K_1 A K_2, \tag{3}$$

where $K_1, K_2 \in \mathrm{e}^{\mathcal{K}}$ and $A \in \mathrm{e}^{\mathcal{H}}$. This decomposition is known as the KAK decomposition of the Lie group $\mathrm{e}^{\mathcal{L}}$.

Cartan classified all the Cartan decompositions of the classical Lie algebras [6]. In particular, up to conjugacy, there exist three types of Cartan decomposition of the special unitary Lie algebra $\mathfrak{su}(n)$, the Lie algebra of skew-Hermitian matrices with zero trace. The decompositions are classified as AI, AII and AIII.

A decomposition of type AI is the Cartan decomposition of $\mathfrak{su}(n)$ into purely real and purely imaginary matrices, i.e.,

$$\mathfrak{su}(n) = \mathfrak{so}(n) \oplus \mathfrak{so}(n)^{\perp}. \tag{4}$$

The orthogonality is given by the inner product $\langle A, B \rangle = \mathrm{tr}(AB^{\dagger})$ where $A, B \in \mathfrak{su}(n)$. The diagonal matrices in $\mathfrak{so}(n)^{\perp}$ span a maximal Abelian subalgebra, so the rank of the decomposition is $n - 1$.

A decomposition of type AII is of the form

$$\mathfrak{su}(2n) = \mathfrak{sp}(n) \oplus \mathfrak{sp}(n)^{\perp}, \tag{5}$$

where $\mathfrak{sp}(n)$ is the Lie algebra of symplectic matrices, namely the subalgebra of $\mathfrak{su}(2n)$ of matrices A satisfying

$$AJ + JA^T = 0,$$

in which J is the $2n \times 2n$ matrix

$$J := \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}.$$

Here and in the rest of this paper, we denote by $\mathbf{1}_n$ the $n \times n$ identity matrix. The rank of the decomposition of type AII is again $n - 1$.

A decomposition of type AIII is defined in terms of two positive integers p and q with $p + q = n$. The decomposition is

$$\mathfrak{su}(n) := \mathcal{K} \oplus \mathcal{P}, \tag{6}$$

where $\mathcal{K}$ is spanned by block diagonal matrices

$$F := \begin{pmatrix} X_{p \times p} & 0 \\ 0 & Y_{q \times q} \end{pmatrix}, \tag{7}$$

with $X_{p \times p}$ and $Y_{q \times q}$ skew-Hermitian and $\mathrm{tr}(X_{p \times p}) + \mathrm{tr}(Y_{q \times q}) = 0$. The rank of this decomposition is $\min\{p, q\}$.

Each Cartan decomposition of $\mathfrak{su}(n)$ is conjugate to one of the decompositions of type AI, AII and AIII. In other words, if $\mathfrak{su}(n) = \mathcal{K} \oplus \mathcal{P}$ is a Cartan decomposition of $\mathfrak{su}(n)$, there exists a unitary matrix T such that $\mathfrak{su}(n) = T\mathcal{K}T^{\dagger} \oplus T\mathcal{P}T^{\dagger}$ is in one of the forms AI, AII

and AIII. These decompositions can be expressed in forms of interest in various contexts, for example with matrices expressed as tensor products of operators on single subsystems in a multipartite quantum system.

In the following, we shall find it convenient to extend these decompositions to decompositions of $\mathfrak{u}(n) = \mathfrak{su}(n) \oplus \mathrm{span}\{i\mathbf{1}_n\}$, the Lie algebra of $\mathbf{U}(n)$. Consider a Cartan decomposition of the special unitary Lie algebra $\mathfrak{su}(n)$ of type either AI or AII. Since the identity matrix $\mathbf{1}_n$ commutes with each element of $\mathfrak{su}(n)$, the Cartan decompositions of $\mathfrak{su}(n)$ of types AI (4) and AII (5) can be naturally extended to decompositions of $\mathfrak{u}(n)$ by replacing $\mathcal{P}$ with $\mathcal{P} \oplus \mathrm{span}\{i\mathbf{1}_n\}$. We also denote these decompositions of types AI and AII. In both Cartan decompositions, the rank becomes n. For decompositions of type AIII, we find it convenient to include $\mathrm{span}\{i\mathbf{1}_n\}$ in the Lie algebra part and replace $\mathcal{K}$ with $\mathcal{K} \oplus \mathrm{span}\{i\mathbf{1}_n\}$, so as to lift the restriction $\mathrm{tr}(X_{p \times p}) + \mathrm{tr}(Y_{q \times q}) = 0$ in (7).

2.2. Cartan decompositions for multipartite quantum systems: CCD and OED

For a multipartite quantum system with N subsystems of dimensions $n_1, n_2, \ldots, n_N$, the set of possible Hamiltonians is the Jordan algebra $i\mathfrak{u}(n_1 n_2 \ldots n_N)$ of $n_1 n_2 \ldots n_N \times n_1 n_2 \ldots n_N$ Hermitian matrices. The Lie algebra associated with the dynamics is $\mathfrak{u}(n_1 n_2 \ldots n_N)$. Cartan decompositions of $\mathfrak{u}(n_1 n_2 \ldots n_N)$ result in decompositions of the corresponding unitary group of quantum evolutions $\mathbf{U}(n_1 n_2 \ldots n_N)$.

The concurrence canonical decomposition (CCD) was studied in [1, 2] as a means of decomposing the dynamics of N two-level systems, into one factor which preserves the concurrence of the density matrix and one factor which does not. It is constructed as follows:

Recall that the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

together with the 2×2 identity matrix $\mathbf{1}_2$ form a basis of the Jordan algebra $i\mathfrak{u}(2)$. An orthogonal basis of $\mathfrak{u}(2^N)$ is given by the tensor products of the form $i\sigma_1 \otimes \cdots \otimes \sigma_N$, where $\sigma_j = \sigma_{x,y,z}$ or $\sigma_j = \mathbf{1}_2$ for all $1 \leqslant j \leqslant N$. Let us denote by $i\mathcal{I}_o$ and $i\mathcal{I}_e$ the respective subspaces of $\mathfrak{u}(2^N)$ spanned by elements of the form $i\sigma_1 \otimes \cdots \otimes \sigma_N$ with an odd or even number of factors σ_j given by Pauli matrices, and the remaining factors equal to the identity $\mathbf{1}_2$. The CCD is the decomposition

$$\mathfrak{u}(2^N) = i\mathcal{I}_o \oplus i\mathcal{I}_e \tag{8}$$

of $\mathfrak{u}(2^N)$. The Lie subgroup $e^{i\mathcal{I}_o}$ associated with the subalgebra $i\mathcal{I}_o$ is a subgroup of $\mathbf{U}(2^N)$ containing all the local transformations. For each $X \in \mathbf{U}(2^N)$ the decomposition (2) holds with $K \in e^{i\mathcal{I}_o}$ and $P = e^R$ with $R \in i\mathcal{I}_e$. The factor K and in particular, any local transformation, does not modify the N-qubit concurrence [1]. Such a decomposition is of type AI if N is even, and of type AII if N is odd.

The odd–even decomposition (OED) was introduced in [4] as a generalization of the CCD to multipartite systems of arbitrary dimensions. The main idea is to construct a decomposition for the whole Lie algebra $\mathfrak{u}(n_1 n_2 \ldots n_N)$ by combining decompositions for the Lie algebras associated with the single subsystems $\mathfrak{u}(n_j)$, $j = 1, \ldots, N$. This is based on the following observation for the CCD. When writing

$$\mathfrak{u}(2) = \mathrm{span}\{i\sigma_x, i\sigma_y, i\sigma_z\} \oplus \mathrm{span}\{i\mathbf{1_n}\},$$

we perform a (trivial) AII decomposition of $\mathfrak{u}(2)$, since $\mathfrak{su}(2) = \mathfrak{sp}(1)$. In the CCD, we collect (modulo i) tensor products with an odd number of elements in the Lie algebra in $\mathcal{I}_o$ and tensor

products with an even number of elements in $\mathcal{I}_e$. The OED [4] is obtained by applying this idea to general Lie algebras $\mathfrak{u}(n_j)$, $j = 1, \ldots, N$. By writing

$$\mathfrak{u}(n_j) = \mathcal{K} \oplus \mathcal{P},$$

with $\mathcal{K}$ conjugate to $\mathfrak{so}(n_j)$ or $\mathfrak{sp}(\frac{n_j}{2})$, and $\mathcal{P} = \mathcal{K}^\perp$, we obtain a decomposition of type AI or AII, respectively. Denoting by σ a generic element of $i\mathcal{K}$ and by S a generic element of $i\mathcal{P}$, we define $\tilde{\mathcal{I}}_o$ and $\tilde{\mathcal{I}}_e$ to be the respective vector space spanned by tensor products of matrices of the type σ and S with an odd or even number of σ terms. The decomposition

$$\mathfrak{u}(n_1 n_2 \ldots n_N) = i\tilde{\mathcal{I}}_o \oplus i\tilde{\mathcal{I}}_e \tag{9}$$

is a Cartan decomposition called the OED. The subspace $i\tilde{\mathcal{I}}_o$ is the Lie subalgebra. This is a generalization of the CCD not only because it applies to systems of arbitrary dimensions, but also because, for every subsystem, we can perform different decompositions of type AI or AII. The CCD is obtained as a special case of the OED (9) when all the subsystems are of dimension 2 and a decomposition of type AII is performed on each subsystem. Generalizing the result on the nature of the CCD decomposition, the OED decomposition is of type AII if an odd number of AII decompositions are performed. Otherwise, it is of type AI. As the CCD is related to the concurrence on N qubits, the OED has the same meaning for the generalized concurrences studied by Uhlmann in [10].

We refer to [3] for a detailed discussion of the CCD and OED decompositions, and to [6] for the mathematical foundations of the Cartan decompositions.

Remark 2.1. The procedure described for the OED allows one great flexibility in the construction of various Cartan decompositions. Not only is one free to choose decompositions of type AI or AII for each subsystem, but one can also choose among the different types of conjugate AI or AII decompositions for each subsystem. This gives a method for the construction of an infinite number of decompositions in terms of tensor product matrices, even for the simplest case of N qubits. This flexibility is crucial in the construction of gradings for the Lie algebra $\mathfrak{u}(n_1 n_2 \ldots n_N)$, and of recursive decompositions, as we shall see in the following two sections.

We observe here that the procedure followed to construct the OED decomposition, applying decompositions of type AI and AII, can be used with few changes to obtain an overall decomposition starting from decompositions of type AIII. More specifically, we perform decompositions of type AIII on each subsystem and collect in the respective subspaces $i\tilde{\mathcal{I}}_o$ and $i\tilde{\mathcal{I}}_e$ the linear combinations of tensor products with an odd or even number of factors in the subalgebra part (modulo i). We again consider the decomposition of $\mathfrak{u}(n_1 n_2 \ldots n_N)$ in (9) but with $i\tilde{\mathcal{I}}_o$ and $i\tilde{\mathcal{I}}_e$ defined in terms of type AIII decompositions.

Theorem 2.1. *Consider the decomposition (9) obtained with decompositions of type AIII as described above. This is a type AIII decomposition of the overall Lie algebra $u(n_1 n_2 \ldots n_N)$. If N is odd, then $i\tilde{\mathcal{I}}_o$ is the Lie subalgebra in the decomposition. If N is even, then $i\tilde{\mathcal{I}}_e$ is the Lie subalgebra in the decomposition.*

Proof. The proof is by induction on N. If $N = 1$ the statement is obvious. Assume the statement is true for $N - 1$, and assume to be concrete that N is odd (exactly the same proof holds for N even). Denote by $\tilde{\mathcal{I}}_o^j$ and $\tilde{\mathcal{I}}_e^j$ the respective spaces of matrices in $i\mathfrak{u}(n_1 n_2 \ldots n_j)$ that are linear combinations of an odd or even number of matrices in $i\mathcal{K}$, where $\mathcal{K}$ is the subalgebra of the AIII decomposition (possibly different for the different subsystems). Let us

denote by $\mathcal{K}$ the subalgebra of (block diagonal) matrices of the AIII decomposition on the last subsystem and by $\mathcal{P}$ its orthogonal complement. We have

$$\tilde{\mathcal{I}}_o^N = \left(\tilde{\mathcal{I}}_e^{N-1} \otimes \mathrm{i}\mathcal{K}\right) \oplus \left(\tilde{\mathcal{I}}_o^{N-1} \otimes \mathrm{i}\mathcal{P}\right), \qquad \tilde{\mathcal{I}}_e^N = \left(\tilde{\mathcal{I}}_e^{N-1} \otimes \mathrm{i}\mathcal{P}\right) \oplus \left(\tilde{\mathcal{I}}_o^{N-1} \otimes \mathrm{i}\mathcal{K}\right).$$

By the inductive assumption, there exists a unitary matrix T in $\mathbf{U}(n_1 n_2 \ldots n_{N-1})$ such that $T^\dagger \tilde{\mathcal{I}}_e^{N-1} T$ is the same as the space of $n_1 n_2 \ldots n_{N-1} \times n_1 n_2 \ldots n_{N-1}$ Hermitian block-diagonal matrices, and $T^\dagger \tilde{\mathcal{I}}_o^{N-1} T$ is the same as the space of $n_1 n_2 \ldots n_{N-1} \times n_1 n_2 \ldots n_{N-1}$ Hermitian block-antidiagonal matrices. Let $T_1 = T^\dagger \otimes \mathbf{1}_{n_N}$, then the subspace $T_1^\dagger \tilde{\mathcal{I}}_o^N T_1$ is spanned by all the matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \otimes \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & F \\ -F^\dagger & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & G \\ -G^\dagger & 0 \end{pmatrix}.$$

The sizes of the matrices A, B, C, D, F and G depend on the indices p, q of the two decompositions. Using corollary 4.3.10 of [15] one can construct a permutation similarity matrix T_2 so that the subspace $(T_1 T_2)^\dagger \tilde{\mathcal{I}}_o^N (T_1 T_2)$ is spanned by all the matrices of the form

$$T_2^\dagger \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \otimes \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} T_2 = \begin{pmatrix} C \otimes A & 0 & 0 & 0 \\ 0 & D \otimes A & 0 & 0 \\ 0 & 0 & C \otimes B & 0 \\ 0 & 0 & 0 & D \otimes B \end{pmatrix},$$

and by all the matrices of the form

$$T_2^\dagger \begin{pmatrix} 0 & F \\ -F^\dagger & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & G \\ -G^\dagger & 0 \end{pmatrix} T_2 = \begin{pmatrix} 0 & 0 & 0 & G \otimes F \\ 0 & 0 & -G^\dagger \otimes F & 0 \\ 0 & -G \otimes F^\dagger & 0 & 0 \\ G^\dagger \otimes F^\dagger & 0 & 0 & 0 \end{pmatrix}.$$

Finally, the conjugation $P \to T_3^\dagger P T_3$, where T_3 has the form

$$T_3 = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & 0 \end{pmatrix}$$

with identity matrices $\mathbf{1}$ of appropriate dimensions, transforms the subspace $(T_1 T_2)^\dagger \tilde{\mathcal{I}}_o^N (T_1 T_2)$ into the standard block diagonal form (7) of the type AIII decomposition. Therefore the subspace $R^\dagger \tilde{\mathcal{I}}_o^N R$ is of form (7) where $R := T_1 T_2 T_3$. It can be verified that R also transforms $\tilde{\mathcal{I}}_e^N$ into the standard block anti-diagonal form of the AIII decomposition. $\qquad \square$

Remark 2.2. The indices p and q of the resulting AIII decompositions of theorem 2.1 are $p = n_1 n_2 \ldots n_{l-1} p_l n_{l+1} \ldots n_N$ and $q = n_1 n_2 \ldots n_{l-1} q_l n_{l+1} \ldots n_N$, where l may refer to any of the subsystems $l = 1, \ldots, N$, and p_l and q_l are the indices of the AIII decomposition of the lth system. The theorem also indicates the inductive construction of the matrix conjugation which maps the AIII decomposition into the standard form. This is of interest for practical computation of the decomposition, as most of the existing numerical algorithms refer to the standard form (6), (7). Note that decompositions constructed by mixing AI or AII decompositions with AIII type decompositions do not give rise to Cartan decompositions.

2.3. Recursive decompositions

A recursive procedure to decompose unitary evolutions into local and entangling factors for the case of *N* qubits was introduced by N Khaneja and S Glaser in [7]. One starts with the Cartan decomposition

$$\mathfrak{su}(2^N) = \mathcal{K} \oplus \mathcal{P} \tag{10}$$

of type AIII, where

$$\mathcal{K} = \text{span}\{\mathbf{1}_2 \otimes A, \sigma_z \otimes B \mid A \in \mathfrak{su}(2^{N-1}), B \in \mathfrak{u}(2^{N-1})\}, \tag{11}$$

$$\mathcal{P} = \text{span}\{\sigma_{x,y} \otimes C \mid C \in \mathfrak{u}(2^{N-1})\}. \tag{12}$$

This allows one to write each special unitary evolution $X \in \mathbf{SU}(2^N)$ as $X = K_1 A K_2$, where $K_1, K_2 \in e^{\mathcal{K}}$ and $A \in e^{\mathcal{A}}$, with $\mathcal{A}$ the Cartan subalgebra contained in $\mathcal{P}$. The subalgebra $\mathcal{K}$ is the direct sum of $\text{span}\{i\sigma_z \otimes \mathbf{1}_{2^{N-1}}\}$ and two copies of $\mathfrak{su}(2^{N-1})$ which form a semisimple Lie algebra. Thus we can again apply Cartan's theorem to further factorize $K_1, K_2 \in e^{\mathcal{K}}$. This is obtained through the decomposition $\mathcal{K} = \mathcal{K}' \oplus \mathcal{P}'$, with

$$\mathcal{K}' = \text{span}\{\mathbf{1}_2 \otimes A \mid A \in \mathfrak{su}(2^{N-1})\}, \tag{13}$$

$$\mathcal{P}' = \text{span}\{\sigma_z \otimes B \mid B \in \mathfrak{u}(2^{N-1})\}, \tag{14}$$

to decompose each $K_1, K_2 \in e^{\mathcal{K}}$, thereby refining the decomposition of *X*. The key observation is that $\mathcal{K}'$ and $\mathfrak{su}(2^{N-1})$ are isomorphic, hence the procedure can be repeated by replacing *N* with $N - 1$.

Another recursive procedure to decompose unitary evolutions was introduced by D D'Alessandro and R Romano in [5]. Such a decomposition applies to bipartite systems of arbitrary dimensions.[1] In the first step, one starts with an OED decomposition using AI types of decomposition on both subsystems, so that $\mathfrak{u}(n_1 n_2)$ is decomposed as in (9), that is, $\mathfrak{u}(n_1 n_2) = i\tilde{\mathcal{I}}_o \oplus i\tilde{\mathcal{I}}_e$, where

$$\tilde{\mathcal{I}}_o := \text{span}\{\sigma \otimes S, S \otimes \sigma\} \tag{15}$$

conjugate to $\mathfrak{so}(n_1 n_2)$. As $\mathfrak{so}(n_1 n_2)$ is also semisimple, one then introduces a Cartan decomposition of $\tilde{\mathcal{I}}_o$ by separating block diagonal and anti-diagonal elements (for two arbitrary indices) in the factors of the basis of $\tilde{\mathcal{I}}_o$. In particular, one writes

$$i\tilde{\mathcal{I}}_o = \mathcal{K} \oplus \mathcal{P},$$

where

$$\mathcal{K} := \text{span}\{i\sigma^D \otimes S^D, iS^D \otimes \sigma^D, i\sigma^A \otimes S^A, iS^A \otimes \sigma^A\}$$

and

$$\mathcal{P} := \text{span}\{i\sigma^D \otimes S^A, iS^D \otimes \sigma^A, i\sigma^A \otimes S^D, iS^A \otimes \sigma^D\},$$

the superscripts *A* and *D* standing for block-antidiagonal and block-diagonal respectively. The Lie algebra $\mathcal{K}$ is isomorphic to the semisimple direct sum $\mathfrak{so}(r) \oplus \mathfrak{so}(f)$ with $r + f = n_1 n_2$. One decomposes $\mathcal{K}$ as

$$\mathcal{K} = \mathcal{K}' \oplus \mathcal{P}',$$

with $\mathcal{K}' := \text{span}\{i\sigma^D \otimes S^D, iS^D \otimes \sigma^D\}$ and $\mathcal{P}' := \text{span}\{i\sigma^A \otimes S^A, iS^A \otimes \sigma^A\}$. The Lie subalgebra $\mathcal{K}'$ is isomorphic to the direct sum of four subalgebras $\mathfrak{so}(r_1) \oplus \mathfrak{so}(r_2) \oplus \mathfrak{so}(r_3) \oplus \mathfrak{so}(r_4)$. Each of the summands is spanned by tensor products of the type in (15) with matrices σ and *S*, where only one sub-block is different from zero. One then iterates the procedure. We refer to [5] for details.

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[1] Extensions to the general multipartite case can be obtained at the price of some notational complexity.

3. Lie algebra grading and recursive Cartan decompositions

In this section, we give the definition of a grading of a Lie algebra and relate a recursive Lie algebra decomposition to a grading. Our goal is to cast recursive decompositions of the unitary group into a general framework. In fact, in the following section we will show that known recursive decompositions, such as those of Khaneja and Glaser [7] and D'Alessandro and Romano [5] reviewed in the previous section, can be obtained from an appropriate grading. We shall also see in the following section how new decompositions can be generated with the procedure described here.

Definition 3.1. *Let $\mathcal{L}$ be a Lie algebra and let M be an index set which has the structure of an additive semigroup. A direct sum decomposition*

$$\mathcal{L} = \bigoplus_{i \in M} \mathcal{L}_i$$

is called an M-grading of $\mathcal{L}$ if the subspaces $\mathcal{L}_i$ and $\mathcal{L}_j$ satisfy the commutation relation $[\mathcal{L}_i, \mathcal{L}_j] \subseteq \mathcal{L}_{i+j}$ for all $i, j \in M$.

In the special case where M is a monoid, that is, a semigroup with an identity element 0, the subspace $\mathcal{L}_0$ is a Lie subalgebra, since it satisfies the commutation relation $[\mathcal{L}_0, \mathcal{L}_0] \subseteq \mathcal{L}_0$.

Example 3.1. Consider the special linear Lie algebra $\mathfrak{sl}(2)$ of 2×2 traceless matrices spanned by

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with the commutation relations $[h, x] = 2x, [x, y] = h$ and $[h, y] = -2y$. Let $M = \{-1, 0, 1\}$. Then M becomes a monoid with addition given by the following table:

$(M, +)$	0	-1	1
0	0	-1	1
-1	-1	0	0
1	1	0	0

The choice of $\mathcal{L}_{-1} = \mathrm{span}\{x\}$, $\mathcal{L}_0 = \mathrm{span}\{h\}$ and $\mathcal{L}_1 = \mathrm{span}\{y\}$ makes $\mathfrak{sl}(2)$ into an M-graded Lie algebra.

A fundamental observation for what follows is that a Cartan decomposition (1) defines a $\mathbb{Z}_2$-grading of the Lie algebra $\mathcal{L}$ with $\mathcal{K} := \mathcal{L}_0$ and $\mathcal{P} := \mathcal{L}_1$.

As we have seen above, for a Lie algebra $\mathcal{L}$, there are many Cartan decompositions. The following proposition shows that p Cartan decompositions give a $\mathbb{Z}_2^p$-grading for general p.

Proposition 3.1. *Consider p Cartan decompositions $\mathcal{L} = \mathcal{L}_0^j \oplus \mathcal{L}_1^j$ for $j = 1, \ldots, p$. Define*

$$\mathcal{L}_{k_1 k_2 \ldots k_p} := \bigcap_{j=1,\ldots,p} \mathcal{L}_{k_j}^j \tag{16}$$

for $k_j \in \mathbb{Z}_2$. Then the vector space decomposition

$$\mathcal{L} = \bigoplus \mathcal{L}_{k_1 k_2 \ldots k_p} \tag{17}$$

forms a $\mathbb{Z}_2^p$-grading of $\mathcal{L}$.

Proof. The proof is by induction on p. The claim is true for $p = 1$. Assume the claim is true for $p - 1$. Let $A \in \mathcal{L}_{k_1 k_2 \ldots k_p}$ and $B \in \mathcal{L}_{l_1 l_2 \ldots l_p}$ where $k_i, l_i \in \mathbb{Z}_2$ with $1 \leqslant i \leqslant p$. Then

it follows that $A \in \mathcal{L}_{k_1 k_2 \dots k_{p-1}}$, $A \in \mathcal{L}_{k_p}$, $B \in \mathcal{L}_{l_1 l_2 \dots l_{p-1}}$, and $B \in \mathcal{L}_{l_p}$. Since $\mathcal{L}$ is both $\mathbb{Z}_2^{p-1}-$graded and $\mathbb{Z}_2-$ graded, we have

$$[A, B] \in \mathcal{L}_{(k_1+l_1)(k_2+l_2)\dots(k_{p-1}+l_{p-1})} \qquad \text{and} \qquad [A, B] \in \mathcal{L}_{(k_p+l_p)}.$$

This implies that

$$[A, B] \in \mathcal{L}_{(k_1+l_1)(k_2+l_2)\dots(k_{p-1}+l_{p-1})} \cap \mathcal{L}_{(k_p+l_p)} = \mathcal{L}_{(k_1+l_1)(k_2+l_2)\dots(k_p+l_p)}$$

and therefore

$$[\mathcal{L}_{k_1 k_2 \dots k_p}, \mathcal{L}_{l_1 l_2 \dots l_p}] \subseteq \mathcal{L}_{(k_1+l_1)(k_2+l_2)\dots(k_p+l_p)}.$$

In conclusion, (17) is a $\mathbb{Z}_2^p$-grading of $\mathcal{L}$. $\square$

Thus a Cartan decomposition of a Lie algebra is a $\mathbb{Z}_2$-grading. A combination of p Cartan decompositions gives a $\mathbb{Z}_2^p$-grading. In order to cast a recursive decomposition in the framework of Lie algebra gradings, we give the following definition.

Definition 3.2. *A recursive decomposition of a Lie algebra $\mathcal{L}$ consists of two sequences of subspaces of $\mathcal{L}$,*

$$\mathcal{S}_0 := \{\mathcal{L}_0, \mathcal{L}_{00}, \mathcal{L}_{000}, \dots, \mathcal{L}_{0^p}\} \qquad and \qquad \mathcal{S}_1 := \{\mathcal{L}_1, \mathcal{L}_{01}, \mathcal{L}_{001}, \dots, \mathcal{L}_{0^{p-1}1}\},$$

both of length p, such that

$$\mathcal{L}_{0^j} = \mathcal{L}_{0^{j+1}} \oplus \mathcal{L}_{0^j 1}$$

is a Cartan decomposition of $\mathcal{L}_{0^j}$ for each $j = 0, \dots, p-1$. That is,

$$[\mathcal{L}_{0^{j+1}}, \mathcal{L}_{0^{j+1}}] \subseteq \mathcal{L}_{0^{j+1}}, \qquad [\mathcal{L}_{0^{j+1}}, \mathcal{L}_{0^j 1}] \subseteq \mathcal{L}_{0^j 1}, \qquad [\mathcal{L}_{0^j 1}, \mathcal{L}_{0^j 1}] \subseteq \mathcal{L}_{0^{j+1}}.$$

Here we have set $\mathcal{L}_{0^0} := \mathcal{L}$ and $\mathcal{L}_{0^0 1} := \mathcal{L}_1$.

Once one has a recursive decomposition of a Lie algebra $\mathcal{L}$, in the sense of the above definition, one can obtain a decomposition of the connected Lie group $\mathrm{e}^{\mathcal{L}}$ associated with $\mathcal{L}$. This is obtained by repeated use of the Cartan decomposition theorem. Assume that $\mathcal{L}$ is semisimple, and that all of the $\mathcal{L}_{0^j}$, $j = 1, \dots, p-1$, are also semisimple. One first writes each element X of $\mathrm{e}^{\mathcal{L}}$ as

$$X = K_1 A K_2,$$

where K_1 and K_2 are in $\mathrm{e}^{\mathcal{L}_0}$, while A belongs to the connected Lie group corresponding to a maximal Abelian subalgebra contained in $\mathcal{L}_1$. Then one applies the Cartan decomposition of $\mathcal{L}_0$ in order to decompose K_1 and K_2, and so on. The resulting decomposition contains several factors.

A $\mathbb{Z}_2^p$-grading of $\mathcal{L}$ induces a recursive decomposition of $\mathcal{L}$ of length p.

Proposition 3.2. *Consider a $\mathbb{Z}_2^p$-grading $\mathcal{L} = \bigoplus \mathcal{R}_{j_1, \dots, j_p}$ of $\mathcal{L}$. Then the sequences $\mathcal{S}_0 := \{\mathcal{L}_{0^k}\}$ and $\mathcal{S}_1 := \{\mathcal{L}_{0^{k-1}1}\}$, defined by*

$$\mathcal{L}_{0^k} := \bigoplus \mathcal{R}_{0^k, j_{k+1}, \dots, j_p} \qquad and \qquad \mathcal{L}_{0^{k-1}1} := \bigoplus \mathcal{R}_{0^{k-1}1, j_{k+1}, \dots, j_p}$$

for $k = 1, \dots, p$, yield a recursive decomposition of $\mathcal{L}$ of length p.

Remark 3.1. Given a recursive decomposition sequence as in definition 3.2, the semisimplicity of the subalgebras $\mathcal{L}_{0^k}$ (for $k = 0, \dots, p$) has to be verified independently. Even in the main case considered here, where the recursive decomposition sequence is obtained from a Lie algebra grading by means of combined Cartan decompositions as in proposition 3.1, semisimplicity is not guaranteed. For example, by combining type AI and AII decompositions

of $\mathfrak{su}(4)$ in the standard basis, one obtains $\mathcal{L}_{00} = \mathfrak{sp}(2) \cap \mathfrak{so}(4)$. This is not semisimple, having an element which commutes with the whole Lie algebra. If a Lie algebra is the direct sum of a semisimple Lie algebra and an Abelian ideal, the Cartan decomposition theorem can be extended in the same fashion as we extended decompositions of $\mathfrak{su}(n)$ to decompositions of $\mathfrak{u}(n)$ in section 2.1.

4. A scheme for recursive decompositions of U(n)

4.1. Special cases

We now show that the recursive decompositions of Khaneja and Glaser [7] and D'Alessandro and Romano [5], summarized in section 2.3, form a special case of the above procedure. In particular, they are induced by an appropriate grading.

Let us start with the decomposition of Khaneja and Glaser [7]. We construct a Lie algebra grading of $\mathfrak{su}(2^N)$ using the prescription of proposition 3.1. Consider p Cartan decompositions
$$\mathfrak{su}(2^N) = \mathcal{L}_0^j \oplus \mathcal{L}_1^j \tag{18}$$
of $\mathfrak{su}(2^N)$ for $j = 1, \ldots, p$, all of type AIII, where $\mathcal{L}_0^1$ and $\mathcal{L}_1^1$ are respectively equal to $\mathcal{L}_0$ and $\mathcal{L}_1$ in (11) and (12) respectively. Now $\mathcal{L}_0^2$ and $\mathcal{L}_1^2$ are defined in the same way as $\mathcal{L}_0^1$ and $\mathcal{L}_1^1$, except for the fact that σ_x and σ_z are interchanged.[2] Such a decomposition is conjugate to the standard type AIII decomposition, the conjugation having the form $A \rightarrow T \otimes \mathbf{1}_{2^{N-1}} A T^\dagger \otimes \mathbf{1}_{2^{N-1}}$ with T the 2×2 matrix which diagonalizes σ_x. The summands $\mathcal{L}_0^3$ and $\mathcal{L}_1^3$ are given by

$$\mathcal{L}_0^3 = \text{span}\{A \otimes \mathbf{1}_2 \otimes C, B \otimes \sigma_z \otimes D \mid A, B \in \mathfrak{u}(2), C, D \in \mathfrak{u}(2^{N-2}), \text{tr}(A \otimes C) = 0\}, \tag{19}$$
$$\mathcal{L}_1^3 = \text{span}\{E \otimes \sigma_{x,y} \otimes F \mid E \in \mathfrak{u}(2), F \in \mathfrak{u}(2^{N-2})\}. \tag{20}$$

This decomposition is again conjugate to the standard type AIII decomposition under the permutation which exchanges the first and second positions. The decomposition $\mathcal{L}_0^4 \oplus \mathcal{L}_1^4$ is the same as $\mathcal{L}_0^3 \oplus \mathcal{L}_1^3$, except for the fact that the roles of σ_x and σ_z are exchanged. The summands $\mathcal{L}_0^5$ and $\mathcal{L}_1^5$ are defined analogously to $\mathcal{L}_0^3$ and $\mathcal{L}_1^3$, using the third position in place of the second. The same holds for $\mathcal{L}_0^6$ and $\mathcal{L}_1^6$, defined as $\mathcal{L}_0^4$ and $\mathcal{L}_1^4$. In this fashion, one can define $p = 2N - 1$ decompositions[3] and therefore a $\mathbb{Z}_2^p$-grading of $\mathfrak{su}(2^N)$. The corresponding pair of sequences giving the recursive decomposition according to proposition 3.2 is

$$\mathcal{L}_0, \mathcal{L}_1, \text{ same as in (11), and (12)},$$
$$\mathcal{L}_{00} = \text{span}\{\mathbf{1}_2 \otimes A \mid A \in \mathfrak{su}(2^{N-1})\},$$
$$\mathcal{L}_{01} = \text{span}\{\sigma_z \otimes B \mid B \in \mathfrak{u}(2^{N-1})\},$$
$$\mathcal{L}_{000} = \text{span}\{\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes C, \mathbf{1}_2 \otimes \sigma_z \otimes D \mid C \in \mathfrak{su}(2^{N-2}), D \in \mathfrak{u}(2^{N-2})\},$$
$$\mathcal{L}_{001} = \text{span}\{\mathbf{1}_2 \otimes \sigma_{x,y} \otimes D \mid D \in \mathfrak{u}(2^{N-2})\},$$
$$\vdots$$
$$\mathcal{L}_{0^{2N-3}} = \text{span}\{\mathbf{1}_{2^{N-1}} \otimes F, \mathbf{1}_{2^{N-2}} \otimes \sigma_z \otimes G \mid F \in \mathfrak{su}(2), G \in \mathfrak{u}(2)\},$$
$$\mathcal{L}_{0^{2N-4}1} = \text{span}\{\mathbf{1}_{2^{N-2}} \otimes \sigma_{x,y} \otimes G \mid G \in \mathfrak{u}(2)\},$$
$$\mathcal{L}_{0^{2N-2}} = \text{span}\{\mathbf{1}_{2^{N-1}} \otimes F, F \in \mathfrak{su}(2)\},$$
$$\mathcal{L}_{0^{2N-3}1} = \text{span}\{\mathbf{1}_{2^{N-2}} \otimes \sigma_z \otimes G \mid G \in \mathfrak{u}(2)\},$$
$$\mathcal{L}_{0^{2N-1}} = \text{span}\{\mathbf{1}_{2^{N-1}} \otimes \sigma_z\},$$
$$\mathcal{L}_{0^{2N-2}1} = \text{span}\{\mathbf{1}_{2^{N-1}} \otimes \sigma_{x,y}\}.$$

---

[2] There is nothing special about σ_x here. One could have chosen σ_y instead.
[3] We stop at $p = 2N - 1$ because $\mathcal{L}_{0^p}$ is $\{0\}$.

This sequence of subspaces is the one corresponding to the Khaneja–Glaser decomposition. Note in particular that each Lie subalgebra $\mathcal{L}_{0^k}$ (for $k = 1, \ldots, 2N - 1$) is either semisimple, or the sum of a semisimple Lie algebra and an Abelian (in fact one-dimensional) subalgebra of elements which commute with the whole Lie algebra. Thus the Cartan decomposition theorem applies in each case (cf remark 3.1).

In order to obtain the recursive decomposition corresponding to the decomposition of D'Alessandro and Romano [5], one constructs a grading by combining three types of decomposition:

(1) An OED decomposition with a type AI decomposition on each system;
(2) OED decompositions constructed using type AIII decompositions on each factor as in theorem 2.1;
(3) Type AIII decompositions in the standard form (separating block diagonal and block antidiagonal matrices).

In particular, let $\mathcal{L}_0^1 = i\tilde{\mathcal{I}}_o$ and $\mathcal{L}_1^1 = i\tilde{\mathcal{I}}_o^\perp$, where $\tilde{\mathcal{I}}_o$ is defined in (15). The summands $\mathcal{L}_0^2$ and $\mathcal{L}_1^2$ are the respective subspaces $i\tilde{\mathcal{I}}_e$ and $i\tilde{\mathcal{I}}_o$, referred to in theorem 2.1. The summands $\mathcal{L}_0^3$ and $\mathcal{L}_1^3$ are the $\mathcal{K}$ and $\mathcal{P}$ subspaces of a type AIII decomposition in standard coordinates, with p and q given by $p = p_1 p_2 + p_1 q_2$ and $q = q_1 p_2 + q_1 q_2$. Here $\{p_1, q_1\}$ and $\{p_2, q_2\}$ are the indices for the type AIII decompositions used for $\mathcal{L}_0^2$ and $\mathcal{L}_1^2$. The summands $\mathcal{L}_0^4, \mathcal{L}_1^4, \mathcal{L}_0^5$ and $\mathcal{L}_1^5$ are constructed analogously to $\mathcal{L}_0^2, \mathcal{L}_1^2, \mathcal{L}_0^3$ and $\mathcal{L}_1^3$ respectively, with different indices $\{p_1, q_1\}$ and $\{p_2, q_2\}$. The same holds for $\mathcal{L}_0^6, \mathcal{L}_1^6, \mathcal{L}_0^7$ and $\mathcal{L}_1^7$, and so on. Each time, the indices $\{p_1, q_1\}$ and $\{p_2, q_2\}$ are changed, differing from the previous ones in order to avoid repetition of decompositions. With these decompositions, one can define a grading, and therefore a recursive decomposition. This decomposition corresponds to the one in [5].

4.2. Construction of new recursive decompositions

It follows from the previous discussion that many recursive decompositions of $\mathfrak{u}(n)$ (or $\mathfrak{su}(n)$) and therefore of $\mathbf{U}(n)$ (or $\mathbf{SU}(n)$) can be obtained. Once one has a certain number p of Cartan decompositions, then a $\mathbb{Z}_2^p$-grading and therefore a recursive decomposition can be obtained. We have seen that known recursive decompositions are a special case of this general procedure. Cartan decompositions can be obtained for example by taking one type of decomposition, e.g., AI, and then use various conjugations. When dealing with multipartite systems, it is convenient to have Cartan decompositions given in terms of tensor products of matrices as the CCD and OED described in subsection 2.2.

As an example we construct a new recursive decomposition of evolutions on N qubits here. We consider the following $2N$ decompositions on $\mathfrak{u}(2^N)$.

(1) A CCD decomposition so that $\mathcal{L}_0^1 = i\mathcal{I}_o^N$ and $\mathcal{L}_1^1 = i\mathcal{I}_e^N$, where $\mathcal{I}_o^N \left(\mathcal{I}_e^N \right)$ is the same as $\mathcal{I}_o$ $(\mathcal{I}_e)$ in (8) with the superscript N denoting the number of positions considered;
(2) An OED decomposition with all 'local' decompositions of type AII except the one on the Nth term which is of type AI and of the form

$$\mathfrak{u}(2) = \text{span}\{i\sigma_z\} \oplus \text{span}\{i\mathbf{1_2}, i\sigma_x, i\sigma_y\};$$

For the resulting decomposition, we have

$$\mathcal{L}_0^2 = \text{span}\big\{i\mathcal{I}_e^{N-1} \otimes \sigma_z, i\mathcal{I}_o^{N-1} \otimes \{\sigma_x, \sigma_y, \mathbf{1_2}\}\big\},$$
$$\mathcal{L}_1^2 = \text{span}\big\{i\mathcal{I}_o^{N-1} \otimes \sigma_z, i\mathcal{I}_e^{N-1} \otimes \{\sigma_x, \sigma_y, \mathbf{1_2}\}\big\}.$$

(3) Same as in (2) but with σ_z and σ_x interchanged to define $\mathcal{L}_0^3$ and $\mathcal{L}_1^3$;

(4) Same as in (2) but with the Nth position replaced by $(N-1)$th position to define $\mathcal{L}_0^4$ and $\mathcal{L}_1^4$;

(5) Same as in (4) with σ_x and σ_z interchanged;

(6) $\rightarrow (2N-1)\ldots$ and so on moving toward the first position, alternating decompositions as in (2) and decompositions as in (3);

(2N) Same as in (2) with the first position replacing the Nth one.

Example 4.1. In the case $N = 3$ we have, with σ denoting any possible Pauli matrix,

$\mathcal{L}_0^1 = \text{span}\{i\sigma \otimes \mathbf{1}_2 \otimes \mathbf{1}_2, i\mathbf{1}_2 \otimes \sigma \otimes \mathbf{1}_2, i\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \sigma, i\sigma \otimes \sigma \otimes \sigma\},$

$\mathcal{L}_1^1 = \text{span}\{i\sigma \otimes \sigma \otimes \mathbf{1}_2, i\mathbf{1}_2 \otimes \sigma \otimes \sigma, i\sigma \otimes \mathbf{1}_2 \otimes \sigma, i\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2\},$

$\mathcal{L}_0^2 = \text{span}\{i\sigma \otimes \mathbf{1}_2 \otimes \{\mathbf{1}_2, \sigma_x, \sigma_y\}, i\mathbf{1}_2 \otimes \sigma \otimes \{\mathbf{1}_2, \sigma_x, \sigma_y\}, i\sigma \otimes \sigma \otimes \sigma_z, i\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \sigma_z\},$

$\mathcal{L}_1^2 = \text{span}\{i\sigma \otimes \sigma \otimes \{\mathbf{1}_2, \sigma_x, \sigma_y\}, i\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \{\mathbf{1}_2, \sigma_x, \sigma_y\}, i\sigma \otimes \mathbf{1}_2 \otimes \sigma_z, i\mathbf{1}_2 \otimes \sigma \otimes \sigma_z\},$

$\mathcal{L}_0^3 = \text{span}\{i\sigma \otimes \mathbf{1}_2 \otimes \{\mathbf{1}_2, \sigma_z, \sigma_y\}, i\mathbf{1}_2 \otimes \sigma \otimes \{\mathbf{1}_2, \sigma_z, \sigma_y\}, i\sigma \otimes \sigma \otimes \sigma_x, i\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \sigma_x\},$

$\mathcal{L}_1^3 = \text{span}\{i\sigma \otimes \sigma \otimes \{\mathbf{1}_2, \sigma_z, \sigma_y\}, i\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \{\mathbf{1}_2, \sigma_z, \sigma_y\}, i\sigma \otimes \mathbf{1}_2 \otimes \sigma_x, i\mathbf{1}_2 \otimes \sigma \otimes \sigma_x\},$

$\vdots$

$\mathcal{L}_0^6 = \text{span}\{i\{\mathbf{1}_2, \sigma_x, \sigma_y\} \otimes \sigma \otimes \mathbf{1}_2, i\{\mathbf{1}_2, \sigma_x, \sigma_y\} \otimes \mathbf{1}_2 \otimes \sigma, i\sigma_z \otimes \sigma \otimes \sigma \otimes \sigma, i\sigma_z \otimes \mathbf{1}_2 \otimes \mathbf{1}_2\},$

$\mathcal{L}_1^6 = \text{span}\{i\{\mathbf{1}_2, \sigma_x, \sigma_y\} \otimes \sigma \otimes \sigma, i\{\mathbf{1}_2, \sigma_x, \sigma_y\} \otimes \mathbf{1}_2 \otimes \mathbf{1}_2, i\sigma_z \otimes \sigma \otimes \mathbf{1}_2, i\sigma_z \otimes \mathbf{1}_2 \otimes \sigma\}.$

In the general case, with the decompositions $\mathfrak{u}(2^N) := \mathcal{L}_0^j \oplus \mathcal{L}_1^j$, $j = 1, \ldots, 2N$ one constructs a grading as in proposition 3.1 and a recursive decomposition according to proposition 3.2. The sequences of subspaces associated with the latter are given, for $k = 0, \ldots, N-1$,[4]

$$\mathcal{L}_{0^{2k+1}} = \text{span}\{i\mathcal{I}_o^{N-k} \otimes \mathbf{1}_{2^k}\},$$

$$\mathcal{L}_{0^{2k}1} = \text{span}\{i\mathcal{I}_e^{N-k} \otimes \sigma_z \otimes \mathbf{1}_{2^{k-1}}\},$$

$$\mathcal{L}_{0^{2k+2}} = \text{span}\{i\mathcal{I}_o^{N-k-1} \otimes \mathbf{1}_{2^{k+1}}, i\mathcal{I}_e^{N-k-1} \otimes \sigma_z \otimes \mathbf{1}_{2^k}\},$$

$$\mathcal{L}_{0^{2k+1}1} = \text{span}\{i\mathcal{I}_e^{N-k-1} \otimes \{\sigma_x, \sigma_y\} \otimes \mathbf{1}_{2^k}\}.$$

In order to apply this recursive decomposition for the recursive decomposition of the Lie group $\mathbf{U}(2^N)$ we make the following two remarks.

Remark 4.1. The Lie subalgebra $\mathcal{L}_{0^{2k+1}} = \text{span}\{i\mathcal{I}_o^{N-k} \otimes \mathbf{1}_{2^{N-k}}\}$, $0 \leqslant k \leqslant N-1$, is isomorphic to $i\mathcal{I}_o^{N-k}$ which is conjugate to $\mathfrak{so}(2^{N-k})$ or $\mathfrak{sp}(2^{N-k-1})$ according to whether $N-k$ is even or odd, respectively. Thus, in every case the Lie subalgebra is semisimple. On the other hand, the Lie subalgebra $\mathcal{L}_{0^{2k}} = \text{span}\{i\mathcal{I}_o^{N-k} \otimes \mathbf{1}_{2^k}, i\mathcal{I}_e^{N-k} \otimes \sigma_z \otimes \mathbf{1}_{2^{k-1}}\}$ is isomorphic to $\mathfrak{u}(2^{N-k})$, and the isomorphism is given by the map

$$A \otimes \mathbf{1}_{2^k} \longmapsto A, \qquad B \otimes \sigma_z \otimes \mathbf{1}_{2^{k-1}} \longmapsto B, \tag{21}$$

where $A \in i\mathcal{I}_o^{N-k}$ and $B \in i\mathcal{I}_e^{N-k}$. This is the direct sum of a semisimple Lie algebra and a one-dimensional subspace whose elements all commute with the elements of the Lie algebra. In all cases, the Cartan decomposition theorem applies.

---

[4] If the factors on the left occupy all the N positions in the tensor products, the factors on the right do not appear.

Remark 4.2. In applying Cartan theorem to obtain a KAK decomposition as in (3) we need to identify the rank and a Cartan subalgebra at each step. The decomposition

$$\mathcal{L}_{0^{2k}} = \mathcal{L}_{0^{2k+1}} \oplus \mathcal{L}_{0^{2k}1},$$

with $k = 0, \ldots, N-1$ is a decomposition of type AI or AII (modulo the isomorphism in (21)) of $\mathfrak{u}(2^{N-k})$, according to whether $N-k$ is even or odd, respectively. In the AI case the rank is 2^{N-k}. A maximal Abelian subalgebra is spanned by the subspace

$$\mathcal{H}_{AI} := \mathrm{span}\big\{i\mathcal{H}_{\frac{N-k}{2}} \otimes \sigma_z \otimes \mathbf{1}_{2^{k-1}}\big\}.$$

Here we have used the following notation,

$$\mathcal{H} := \mathrm{span}\{\sigma_x \otimes \sigma_x, \sigma_y \otimes \sigma_y, \sigma_z \otimes \sigma_z, \mathbf{1}_2 \otimes \mathbf{1}_2\}, \tag{22}$$

and $\mathcal{H}_l$ denotes the set obtained by tensor products of l elements of $\mathcal{H}$, that is, $\mathcal{H}_l = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$, l times.[5] In the odd, AII, case, the rank is 2^{N-k-1}. A Cartan subalgebra in this case is given by

$$\mathcal{H}_{AII} := \mathrm{span}\big\{i\mathcal{H}_{\frac{N-k-1}{2}} \otimes \mathbf{1}_2 \otimes \sigma_z \otimes \mathbf{1}_{2^{k-1}}\big\}.$$

The decomposition

$$\mathcal{L}_{0^{2k+1}} = \mathcal{L}_{0^{2k+2}} \oplus \mathcal{L}_{0^{2k+1}1},$$

with $k = 0, \ldots, N-1$, is a decomposition of $\mathfrak{so}(2^{N-k})$ or $\mathfrak{sp}(2^{N-k-1})$ according to whether $N-k$ is even or odd. In the first case, it is a decomposition of type DIII (we refer to [6] for decompositions of Lie algebras different from $\mathfrak{u}(n)$) which has rank 2^{N-k-2}. The Cartan subalgebra can be taken equal to

$$\mathcal{H}_{\mathrm{DIII}} := \mathrm{span}\big\{i\mathcal{H}_{\frac{N-k-2}{2}} \otimes \mathbf{1}_2 \otimes \sigma_x \otimes \mathbf{1}_{2^k}\big\}.$$

In the second case, it is a decomposition of type CI and the associated rank is 2^{N-k-1}. The Cartan subalgebra can be taken equal to

$$\mathcal{H}_{\mathrm{CI}} := \mathrm{span}\big\{i\mathcal{H}_{\frac{N-k-1}{2}} \otimes \sigma_x \otimes \mathbf{1}_{2^k}\big\}.$$

5. An example of computation

In this section, we use an example to discuss some of the computational issues arising in recursive decompositions. In particular, we focus on the application of the recursive procedure described in the previous section to a generalized SWAP operator $X_{\mathrm{sw}} \in \mathbf{U}(8)$. In the tensor product basis, the action of X_{sw} is defined by

$$X_{\mathrm{sw}} : |i\rangle \otimes |j\rangle \otimes |k\rangle \longmapsto |j\rangle \otimes |k\rangle \otimes |i\rangle,$$

---

[5] Using the fact that $\mathcal{H}$ is a commuting set and induction on l along with the formula

$$[K \otimes L, M \otimes N] = [K, M] \otimes (L \cdot N) + (M \cdot K) \otimes [L, N],$$

it is easy to see that $\mathcal{H}_l$ is also a commuting set.

where $i, j, k = 0, 1$, refers to an orthonormal basis $\{|0\rangle, |1\rangle\}$ of the Hilbert space of each of three two level systems. X_{sw} is the cyclic left shift operator acting on three qubits. The matrix representation of this operator is given by

$$X_{sw} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Our goal is to factorize X_{sw} in terms of elementary matrices. Using the Cartan decomposition of the previous section (cf example 4.1), one can construct a grading and therefore obtain a recursive decomposition of $\mathfrak{u}(8)$. Modulo isomorphisms, the sequences characterizing the recursive decomposition are given by

$$\mathcal{S}_0 = \{\mathfrak{sp}(4), \mathfrak{u}(4), \mathfrak{so}(4), \mathfrak{u}(2), \mathfrak{sp}(1), \mathfrak{u}(1)\}, \tag{23}$$

$$\mathcal{S}_1 = \{\mathfrak{sp}(4)^\perp, \mathfrak{u}(4)^\perp, \mathfrak{so}(4)^\perp, \mathfrak{u}(2)^\perp, \mathfrak{sp}(1)^\perp, \mathfrak{u}(1)^\perp\}. \tag{24}$$

Most of the algorithms for the computation of decompositions of the unitary group are given in standard coordinates. To transform the problem into standard coordinates, one uses an orthogonal change of basis. According to [1] the associated matrix is given by

$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix is referred to as the *finagler*. After this change of coordinates, X_{sw} takes the form $\tilde{X}_{sw} = F^T X_{sw} F$, with $\tilde{X}_{sw} = \mathbf{1}_2 \otimes X'_{sw}$ where

$$X'_{sw} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

To perform the decomposition, we follow the sequence of subspaces in (23) and (24). The first step is to compute the decomposition of $\tilde{X}_{sw}$ induced by the Cartan pair $(\mathfrak{sp}(4), \mathfrak{sp}(4)^\perp)$ of $\mathfrak{u}(8)$. It can be verified that $\tilde{X}_{sw}$ is symplectic, i.e., $\tilde{X}_{sw} \in \mathbf{Sp}(4)$, therefore its decomposition is trivial. Moreover, $\tilde{X}_{sw}$ is contained in the image of $\mathbf{U}(4)$ embedded into $\mathbf{Sp}(4)$[6] and represented by X'_{sw} in $\mathbf{U}(4)$. Indeed, X'_{sw} is not only unitary but orthogonal, i.e., $X'_{sw} \in \mathbf{SO}(4)$. Hence the decompositions induced by the Cartan pairs $(\mathfrak{u}(4), \mathfrak{u}(4)^\perp)$, and $(\mathfrak{so}(4), \mathfrak{so}(4)^\perp)$ are also trivial. The computational problem is now to find the decomposition $X'_{sw} = K'_1 A' K'_2$ induced by the

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[6] An embedding of $\mathbf{U}(n)$ into $\mathbf{Sp}(n)$ or $\mathbf{SO}(2n)$ is given by the map $U + iV \mapsto \begin{pmatrix} U & V \\ -V & U \end{pmatrix}$ where U and V are real matrices.

Cartan pair $(\mathfrak{u}(2), \mathfrak{u}(2)^\perp)$ so that K_1' and K_2' are contained in the image of $\mathbf{U}(2)$ embedding into $\mathbf{SO}(4)$, and A' is the exponential of an element of the suitable Cartan subalgebra, i.e., $A' = \mathrm{diag}(E, E^{-1})$. Let us partition X_{sw}' into 2×2 blocks, i.e.,

$$X_{\mathrm{sw}}' = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

Choose $K_2' = \mathbf{1}_4$. Then X_{sw}' decomposes as

$$X_{\mathrm{sw}}' = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & E^{-1} \end{pmatrix}, \tag{25}$$

where $A + \mathrm{i}B \in \mathbf{U}(2)$. This equation is equivalent to two matrix equations

$$X_{11} - \mathrm{i}X_{21} = (A + \mathrm{i}B)E, \qquad X_{22} + \mathrm{i}X_{12} = (A + \mathrm{i}B)E^{-1},$$

which implies that

$$E^2 = (X_{22} + \mathrm{i}X_{12})^{-1}(X_{11} - \mathrm{i}X_{21}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Once E is determined from the last equation, we obtain A and B using (25) so that

$$K_1' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \end{pmatrix}, \qquad A' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}. \tag{26}$$

In the next step, we decompose K_1' using the Cartan pairs $(\mathfrak{sp}(1), \mathfrak{sp}(1)^\perp)$ and $(\mathfrak{u}(1), \mathfrak{u}(1)^\perp)$.

As a final result, we obtain

$$\tilde{X}_{\mathrm{sw}} = \tilde{L}_1 \tilde{L}_2 \tilde{L}_3 \tilde{L}_4, \tag{27}$$

where

$$\tilde{L}_1 = \tfrac{1}{\sqrt{2}}(\mathbf{1}_8 - \mathrm{i}\mathbf{1}_2 \otimes \sigma_y \otimes \mathbf{1}_2), \qquad \tilde{L}_2 = \tfrac{1}{\sqrt{2}}(\mathbf{1}_8 + \mathrm{i}\mathbf{1}_2 \otimes \sigma_y \otimes \sigma_z),$$

$$\tilde{L}_3 = \tfrac{1}{\sqrt{2}}(\mathbf{1}_8 + \mathrm{i}\mathbf{1}_4 \otimes \sigma_y), \qquad \tilde{L}_4 = \mathbf{1}_2 \otimes A',$$

where A' is defined in (26). We map $\tilde{X}_{\mathrm{sw}}$ in (27) back to the tensor product basis to write

$$X_{\mathrm{sw}} = L_1 L_2 L_3 L_4, \tag{28}$$

where $L_k = F \tilde{L}_k F^T$, $1 \leqslant k \leqslant 4$, where F is the finagler defined in (25). Finally, we write all the factors in (28) as exponentials of matrices in the tensor product basis to obtain

$$X_{\mathrm{sw}} = \mathrm{e}^{\frac{-\mathrm{i}\pi}{4}\sigma_y \otimes \sigma_z \otimes \sigma_x}\, \mathrm{e}^{\frac{\mathrm{i}\pi}{4}\sigma_x \otimes \sigma_z \otimes \sigma_y}\, \mathrm{e}^{\frac{\mathrm{i}\pi}{4}\sigma_y \otimes \sigma_x \otimes \sigma_z}\, \mathrm{e}^{\frac{-\mathrm{i}\pi}{4}\sigma_x \otimes \sigma_y \otimes \sigma_z}. \tag{29}$$

For the sake of comparison, we factorize X_{sw} using the decomposition of Khaneja and Glaser [7]. We have shown that this factorization corresponds to the sequences

$$\mathcal{S}_0 = \{\mathcal{L}_0, \mathcal{L}_{0^2}, \mathcal{L}_{0^3}, \mathcal{L}_{0^4}, \mathcal{L}_{0^5}\}, \qquad \mathcal{S}_1 = \{\mathcal{L}_1, \mathcal{L}_{01}, \mathcal{L}_{0^21}, \mathcal{L}_{0^31}, \mathcal{L}_{0^41}\}$$

(cf the elements of $\mathcal{S}_0$ and $\mathcal{S}_1$ in subsection 4.1 with $N = 3$). Recall that $\mathcal{L}_0 = \mathrm{span}\{\mathbf{1} \otimes A, \sigma_z \otimes B \mid A \in \mathfrak{su}(4), B \in \mathfrak{u}(4)\}$. We find it convenient to choose the Cartan subalgebra as the span of matrices of type $\sigma_x \otimes D$ with D diagonal for the Cartan pair $(\mathcal{L}_0, \mathcal{L}_1)$ of $\mathfrak{u}(8)$. Therefore the corresponding decomposition of X_{sw} is given by

$$X_{\mathrm{sw}} = K_1 A K_2, \tag{30}$$

where $K_j = \mathrm{diag}(K_{j1}, K_{j2})$, with K_{jk}, $1 \leqslant j, k \leqslant 2$, 4×4 unitary and $A = \begin{pmatrix} D_1 & D_2 \\ D_2 & D_1 \end{pmatrix}$ where D_j is diagonal with $D_1^2 - D_2^2 = \mathbf{1}_4$. Following the procedure described in [3] (section 8.2.3), we obtain the matrices

$$K_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad K_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad K_{21} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$K_{22} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The next step is to factorize both K_1 and K_2 using the Cartan pair $(\mathcal{L}_{0^2}, \mathcal{L}_{01})$. Note that $\mathcal{L}_0^2$ is given by $\mathbf{1} \otimes \mathfrak{su}(4)$. Choosing the Cartan subalgebra as the span of matrices of type $\sigma_z \otimes D$, with diagonal D, induces the decomposition

$$\begin{pmatrix} K_{j1} & 0 \\ 0 & K_{j2} \end{pmatrix} = \begin{pmatrix} L_{j1} & 0 \\ 0 & L_{j1} \end{pmatrix} \begin{pmatrix} A_j & 0 \\ 0 & A_j^{-1} \end{pmatrix} \begin{pmatrix} L_{j2} & 0 \\ 0 & L_{j2} \end{pmatrix} \tag{31}$$

where $L_{j1}, L_{j_2} \in \mathbf{SU}(4)$ and A_j is diagonal. In order to achieve this decomposition, we set

$$\begin{pmatrix} K_{j1} & 0 \\ 0 & K_{j2} \end{pmatrix} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & P^\dagger \end{pmatrix}$$

with unitary K and P to obtain two matrix equations $K_{j1} = KP$ and $K_{j2} = KP^\dagger$. Then it follows that $P^2 = K_{j2}^\dagger K_{j1}$. We diagonalize P^2 with a unitary matrix U to write $P^2 = U\Lambda U^\dagger$, and we choose $D = \Lambda^{\frac{1}{2}}$ with $\det(D) = 1$ so that $P = UDU^\dagger$. Once P is determined, K can be found from the matrix equation $K_{j2} = KP^\dagger$. Finally we choose $L_{j1} = KU$, $L_{j2} = U^\dagger$ and $A_j = D$ to obtain the desired decomposition (31).

Applying this procedure, we obtain

$$L_{11} = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 & 0 & 1 \\ 0 & -i & 1 & 0 \\ i & 0 & 0 & -1 \\ 0 & i & 1 & 0 \end{pmatrix}, \qquad L_{12} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

$$L_{21} = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 & 0 & -1 \\ 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \end{pmatrix}, \qquad L_{22} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \end{pmatrix},$$

and

$$A_1 = A_2 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Similarly, we repeat the first two steps with the respective Cartan pairs $(\mathcal{L}_{0^3}, \mathcal{L}_{0^21})$ and $(\mathcal{L}_{0^4}, \mathcal{L}_{0^31})$ to decompose L_{jk}, with $1 \leqslant j, k \leqslant 2$. Finally, writing all the factors as exponentials, we obtain the factorization

$$X_{\text{sw}} = K_1 A K_2, \tag{32}$$

with

$$K_1 = \mathrm{e}^{\frac{\mathrm{i}\pi}{4}\mathbf{1}_4\otimes\sigma_z}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{4}\mathbf{1}\otimes\sigma_z\otimes\sigma_z}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{4}\mathbf{1}\otimes\sigma_x\otimes\mathbf{1}}\,\mathrm{e}^{\frac{-\mathrm{i}\pi}{4}\mathbf{1}_4\otimes\sigma_y}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{4}\mathbf{1}_4\otimes\sigma_x}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{4}\mathbf{1}\otimes\sigma_z\otimes\sigma_z}\,\mathrm{e}^{\frac{-\mathrm{i}\pi}{4}\mathbf{1}_4\otimes\sigma_x}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{4}\sigma_z\otimes\mathbf{1}\otimes\sigma_z}$$

$$\times\,\mathrm{e}^{\frac{\mathrm{i}\pi}{4}\sigma_z\otimes\sigma_z\otimes\sigma_z}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{4}\mathbf{1}\otimes\sigma_x\otimes\mathbf{1}}\,\mathrm{e}^{\frac{-\mathrm{i}\pi}{4}\mathbf{1}\otimes\sigma_x\otimes\sigma_z}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{4}\mathbf{1}\otimes\sigma_z\otimes\mathbf{1}}\,\mathrm{e}^{\frac{-\mathrm{i}\pi}{4}\mathbf{1}\otimes\sigma_z\otimes\sigma_z}\,\mathrm{e}^{\frac{3\mathrm{i}\pi}{4}\mathbf{1}_4\otimes\sigma_y},$$

$$A = \mathrm{e}^{\frac{\mathrm{i}\pi}{4}\sigma_x\otimes\sigma_z\otimes\mathbf{1}}\,\mathrm{e}^{\frac{-\mathrm{i}\pi}{4}\sigma_x\otimes\mathbf{1}\otimes\sigma_z},$$

$$K_2 = \mathrm{e}^{\frac{\mathrm{i}\pi}{2}\mathbf{1}_4\otimes\sigma_y}\,\mathrm{e}^{\frac{-\mathrm{i}\pi}{4}\mathbf{1}\otimes\sigma_z\otimes\mathbf{1}}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{4}\mathbf{1}\otimes\sigma_z\otimes\sigma_z}\,\mathrm{e}^{\frac{3\mathrm{i}\pi}{4}\mathbf{1}_4\otimes\sigma_y}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{4}\mathbf{1}\otimes\sigma_x\otimes\mathbf{1}}\,\mathrm{e}^{\frac{-\mathrm{i}\pi}{4}\mathbf{1}\otimes\sigma_x\otimes\sigma_z}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{4}\sigma_z\otimes\mathbf{1}\otimes\sigma_z}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{4}\sigma_z\otimes\sigma_z\otimes\sigma_z}$$

$$\times\,\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\mathbf{1}_4\otimes\sigma_z}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\mathbf{1}\otimes\sigma_z\otimes\sigma_z}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{4}\mathbf{1}\otimes\sigma_x\otimes\mathbf{1}}\,\mathrm{e}^{\frac{-\mathrm{i}\pi}{4}\mathbf{1}\otimes\sigma_x\otimes\sigma_z}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\mathbf{1}_4\otimes\sigma_y}\,\mathrm{e}^{\frac{-\mathrm{i}\pi}{4}\mathbf{1}\otimes\sigma_z\otimes\mathbf{1}}\,\mathrm{e}^{\frac{-\mathrm{i}\pi}{4}\mathbf{1}\otimes\sigma_z\otimes\sigma_z}\,\mathrm{e}^{\frac{-\mathrm{i}\pi}{4}\mathbf{1}_4\otimes\sigma_y},$$

where we used $\mathbf{1}$ to denote $\mathbf{1}_2$.

6. Conclusions

Grading of a Lie algebra, Cartan decompositions and recursive decompositions of a Lie group are interrelated ideas. From a set of p Cartan decompositions, one can naturally obtain a $\mathbb{Z}_2^p$-grading of a Lie algebra and a recursive decomposition of the associated Lie group. Known procedures for the recursive decomposition of the unitary group of quantum evolutions are special cases of this general scheme. When dealing with multipartite quantum systems, it is convenient if the decompositions used in the procedure are given in terms of tensor products of basis elements of the Lie algebras associated with the single subsystems. This is the case for the CCD decomposition on N qubits and the OED decomposition in its various forms. In this way, the factors of each element of the group are exponentials of tensor products, and one can identify local operations as well as multi-body interactions.

We have given a new recursive decomposition applying the general procedure, along with an example of computation (section 5). For this example, formulae (29) and (32), which are obtained applying the results of [7], give different decompositions. In general, different recursive decompositions of $\mathfrak{u}(n)$ will result in different factorizations of $\mathbf{U}(n)$. The framework presented here gives a virtually unbounded number of alternatives to decompose $\mathbf{U}(n)$ and parametrize quantum evolutions. This framework is very general but uses only one type of decompositions of Lie groups, the Cartan decompositions. It is possible that different types of decompositions such as Bruhat and Iwasawa [16] could be used to obtain different schemes.

Acknowledgments

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